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Large Gauge Transformations and Magnetic Vortices in Axial Gauge QCD

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Abstract

It is shown that in the modified axial gauge version of canonically quantized QCD_{3+1} on a torus only nongeneric gauge field configurations allow for large gauge transformations. For the other configurations, the gauge is fixed completely. Such configurations carry nonzero total magnetic abelian fluxes, correspond to magnetic vortices parallel to the coordinate axes and are incorporated using both singular gauge fields and a change of boundary conditions.

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1 Introduction

One of the main issues in nonabelian gauge theories is the presence of redundant variables. Eliminating them by “gauge fixing”, one hopes to identify the relevant degrees of freedom, the nonperturbative part of which may solve the outstanding questions in the low energy regime of these theories. This hope has been fostered recently by lattice calculations [1] in the maximal Abelian gauge [2] hinting on certain singular field configurations which can be interpreted as monopoles to be relevant for confinement and other phenomena in QCD.

In this context, a Hamiltonian formulation of QCD is especially useful since it allows one to bear in mind all intuition and techniques of ordinary quantum mechanics; formulating the theory in terms of unconstrained, “physical” variables is the easiest way to render gauge invariant results in approximations. Amongst others, this has triggered interest into cases, eg. [3]–[8], in which at least a partial elimination of redundant degrees of freedom can be done beyond ordinary perturbation theory in order to obtain a deeper insight into the nonperturbative sector.

The goal of this paper is to identify and interpret physically in a quantum mechanical framework the configurations which are connected to nonperturbative processes, i.e. such in which the vacuum- ϑ -angle is relevant. This is done in a special completely gauge fixed formulation, namely the modified axial gauge [3, 7] in Hamiltonian QCD on a torus T^3 as spatial manifold. Here, in contradistinction to the naive axial gauge $A_3 = 0$, the eigenphases of the Polyakov loop in x_3 -direction are kept as dynamical variables.

Before proceeding, it is useful to recapitulate a general consideration. A gauge fixing is complete if there exists one and only one parametrization of function space, i.e. if the canonical variables are uniquely determined given a complete set of observables [9]. A severe obstacle to a complete elimination of redundant variables is Singer’s theorem [10] stating that there is no local gauge fixing procedure on compact manifolds in nonabelian gauge theories which allows for a continuous choice of exactly one vector potential on each gauge orbit, i.e. that one will in the way of complete gauge fixing always encounter Gribov ambiguities [11] for some field configurations.

A crude, semiclassical argument goes as follows: The Pontryagin index [12]

$$Q := -\frac{g^2}{32\pi^2} \int_{\mathcal{M} \times [0;T]} d^4x \, \varepsilon^{\mu\nu\rho\sigma} \text{tr}[F_{\mu\nu}(x)F_{\rho\sigma}(x)] = \int_{\mathcal{M} \times [0;T]} d^4x \, \partial_\mu K^\mu(x) \quad (1.1)$$

is an arbitrary integer for $SU(N)$ gauge theories whenever the spatial manifold \mathcal{M} is compact and temporal development connects two identical physical situations at times $t = 0, T$, hence effectively compactifies time as well. Field configurations with topological charge $Q \neq 0$ are associated with tunneling processes of finite energy and probability, semiclassically instantons, and are held to be important for the explanation of many aspects of the low energy regime of QCD.

Q is gauge invariant and can be written as the integral over a 4-divergence of a current

$K^\mu(x)$ containing only fields and their derivatives [12], so that

$$Q = \oint_{\partial[\mathcal{M} \times S^1]} d^3 \Sigma_\mu K^\mu - \oint_P d^3 \Sigma_\mu K^\mu . \quad (1.2)$$

The second term on the right hand side takes into account possible singularities at points P , and $\partial[\mathcal{M} \times S^1]$ is the boundary of the largest chart admissible on $\mathcal{M} \times S^1$. Without loss of generality, one can choose boundary conditions on \mathcal{M} so that the spatial integrals of the first term are zero. Because the gauge fixing is complete, $K^\mu(t=0) = K^\mu(t=T)$. Hence the current K^μ together with the gauge field has to be singular at least at one point on $\mathcal{M} \times S^1$ whenever $Q \neq 0$.

This singularity is a coordinate singularity in the sense that its position and nature is a priori arbitrary and depends on the gauge choosen, yet it is indispensable for the description of underlying physics. Mathematically, there is a close connection to the lower dimensional example of Dirac monopoles [12], i.e. QED on S^2 , where the Dirac string starting from the centre of the sphere is seen as a singularity of the gauge field configuration on S^2 which can be rotated to an arbitrary position, but cannot be removed. All observables (like the field energy density) remain finite, and excluding the singularity of the Dirac string from the sphere, one arrives at a field which is regular everywhere on the resulting disk, but has nontrivial boundary conditions in the vicinity of the string, thus rendering nonzero total magnetic flux.

Therefore, in completely gauge fixed formulations it is important to understand how these singularities occur in detail in order to maintain them in approximations aiming at the nonperturbative sector of QCD.

In the Coulomb gauge, Jackiw, Muzinich and Rebbi [13] have shown in a semiclassical context the nature of the singularities and their position at the Gribov horizon [11]; in a modified light cone gauge, Franke, Novozhilov and Prokhvatilov [5] have made similar considerations. Recently, Chernodub and Gubarev [14] have connected instantons with the abelian monopoles in the maximal abelian gauge [2].

The paper is organized as follows: Section 2 will briefly review the gauge fixing process by unitary gauge fixing transformations (UGFT) [15] in canonically quantized QCD as described in [3], stressing the importance of the Polyakov loop and of the Jacobian arising from the coordinate transformation in field space. It ends with a construction of the residual gauge transformations in the physical Hilbert space. Using the standard derivation of the vacuum- ϑ -angle in the Hamiltonian formulation as starting point, section 3 will discuss how the boundary conditions have to be changed in the process of gauge fixing and for which field configurations large gauge transformations may occur. In section 4, the underlying physics is discussed, and the last section presents conclusions and an outlook.

2 QCD Hamiltonian in the Modified Axial Gauge

The Hamiltonian of pure QCD in the Weyl gauge $A_0 = 0$

$$H = \int d^3x \operatorname{tr} [\vec{\Pi}^2(\vec{x}) + \vec{B}^2(\vec{x})] \quad (2.1)$$

$$B_i^a(\vec{x}) = \frac{1}{2}\epsilon_{ijk}F_{jk}^a(\vec{x}) \quad , \quad F_{kl} = \partial_k A_l - \partial_l A_k - ig[A_k, A_l] \quad (2.2)$$

is quantized by imposing the canonical commutation relations

$$[A_k^a(\vec{x}), \Pi_l^b(\vec{y})] = i\delta_{kl}\delta^{ab}\delta^{(3)}(\vec{x} - \vec{y}) \quad (2.3)$$

between fields \vec{A} and momenta $\vec{\Pi}$, where $\vec{D} = \vec{\partial} - ig\vec{A}$ is the covariant derivative and $\mathcal{O}^a = 2\text{tr}[\mathcal{O}t^a]$. t^a are the $N^2 - 1$ hermitean, traceless generators of the Lie algebra of $SU(N)$, $\text{tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$, and t^{a_0} are the $N - 1$ generators of the Cartan subalgebra, i.e. of diagonal matrices.

The Weyl gauge allows for time independent gauge transformations whose infinitesimal generator is Gauß's law. It cannot be derived as an equation of motion in the Hamiltonian formalism but has to be imposed as a constraint on physical states $|\text{phys}\rangle$.

$$[\vec{\partial} \cdot \vec{\Pi}^a(\vec{x}) + gf^{abc}\vec{A}^b(\vec{x}) \cdot \vec{\Pi}^c(\vec{x})] |\text{phys}\rangle = 0 \quad \forall a \quad (2.4)$$

Finite gauge transformations are implemented by the unitary operator

$$\Omega[\beta] := \exp -i \int d^3x \, 2\text{tr} [-\vec{\Pi}(\vec{x}) \cdot \vec{D}(\vec{x})\beta(\vec{x})] \quad , \quad \tilde{V}(\vec{x}) = e^{ig\beta(\vec{x})} \quad (2.5)$$

$$\Omega[\beta] \vec{A}(\vec{x}) \Omega^\dagger[\beta] = \tilde{V}(\vec{x}) \left(\vec{A}(\vec{x}) + \frac{i}{g} \vec{\partial} \right) \tilde{V}^\dagger(\vec{x}) =: \tilde{V} \vec{A}(\vec{x}) \quad .$$

On a torus T^3 , one can without loss of generality impose periodic boundary conditions for all fields and derivatives as well as for the gauge transformations¹

$$\vec{A}(\vec{x}^{(i)}) = \vec{A}(\vec{x}^{(i)} + L\vec{e}_i) \quad , \quad \vec{\Pi}(\vec{x}^{(i)}) = \vec{\Pi}(\vec{x}^{(i)} + L\vec{e}_i) \quad , \quad \tilde{V}(\vec{x}^{(i)}) = \tilde{V}(\vec{x}^{(i)} + L\vec{e}_i) \quad (2.6)$$

where $\vec{x}^{(i)}$ denotes a point with vanishing i th component on the boundary of the corresponding box with length of the edge L .

Following [3], the redundant degrees of freedom can be eliminated via an UGFT. One solves Gauß's law for $\Pi_3(\vec{x})$ (here written symbolically) in the sector of physical states,

$$\Pi_3(\vec{x}) |\text{phys}\rangle = \left[\tilde{p}_3(\vec{x}) - \frac{1}{D_3} \vec{D}_\perp(\vec{x}) \cdot \vec{\Pi}_\perp(\vec{x}) \right] |\text{phys}\rangle \quad . \quad (2.7)$$

Here, $\tilde{p}_3(\vec{x})$ is the zero mode part of $\Pi_3(\vec{x})$ w.r.t. $D_3(\vec{x})$. Performing a coordinate transformation in field space by the unitary “gauge fixing” operator

$$U : \{ \vec{A}, \vec{\Pi} \} \rightarrow \{ \vec{A}', \vec{\Pi}' ; \vec{A}'^{\text{unphys}}, \vec{\Pi}'^{\text{unphys}} \} \quad , \quad (2.8)$$

one induces a “gauge transformation” on the fields²,

$$U \vec{A}(\vec{x}) U^\dagger = \tilde{U} \vec{A}'(\vec{x}) : A_3(\vec{x}) = \tilde{U}(\vec{x}) \left(A'_3(\vec{x}_\perp) + \frac{i}{g} \partial_3 \right) \tilde{U}^\dagger(\vec{x}) \quad , \quad (2.9)$$

¹Since fermions don't affect the arguments given in this paper and only make the formulae more lengthy, I leave them out here. Their sole trace is not to allow for twisted boundary conditions [16].

²Formally, it is only a gauge transformation for the fields \vec{A}_\perp , $\vec{\Pi}_\perp$ and depends on A_3 . The notation for A'_3 is slightly different from the one in [3].

so that the unphysical fields \vec{A}^{unphys} don't occur in the transformed Hamiltonian and the unphysical momenta $\vec{\Pi}^{\text{unphys}}$ are eliminated in the transformed physical Hilbert space by the transformed Gauß's law.

“Zero mode” fields $A'_3(\vec{x}_\perp)$ and conjugate momenta Π'_3 obey the modified axial “gauge condition” [3, 7]

$$A'_3(\vec{x}_\perp), \Pi'_3(\vec{x}_\perp) \text{ diagonal}, \quad \partial_3 A'_3(\vec{x}_\perp) = 0 = \partial_3 \Pi'_3(\vec{x}_\perp) \quad (2.10)$$

and remain relevant degrees of freedom since $A'_3(\vec{x}_\perp)$ are the phases of the gauge invariant eigenvalues $\exp igLA'_3(\vec{x}_\perp)$ of the Polyakov loop

$$\text{Pexp} ig \int_0^L dx_3 A_3(\vec{x}) . \quad (2.11)$$

This means that transverse, colour neutral gluons moving in the (x_1, x_2) -plane with polarization in the x_3 -direction remain physical. It is also expressed in the fact that the solution to (2.9) cannot be given for $A'_3(\vec{x}) = 0$ since then \tilde{U} would not be periodic in x_3 -direction. Allowing for a colour neutral zero mode, one finds

$$\tilde{U}(\vec{x}) = \text{Pe}^{ig \int_0^{x_3} dy_3 A_3(\vec{x}_\perp, y_3)} e^{ig\Delta(\vec{x}_\perp)} e^{-igx_3 A'_3(\vec{x}_\perp)}, \quad (2.12)$$

where $e^{ig\Delta(\vec{x}_\perp)}$ diagonalizes the Polyakov loop (2.11)

$$e^{ig\Delta(\vec{x}_\perp)} e^{igLA'_3(\vec{x}_\perp)} e^{-ig\Delta(\vec{x}_\perp)} = \text{Pexp} ig \int_0^L dx_3 A_3(\vec{x}) . \quad (2.13)$$

In the space of transformed physical states

$$|\text{phys}'[A']\rangle := \sqrt{\mathcal{J}[A'_3]} U[A_3] |\text{phys}[A]\rangle, \quad (2.14)$$

the resulting Hamiltonian and canonical commutation relations are [3]

$$\begin{aligned} \langle \text{phys}'_1 | H' | \text{phys}'_2 \rangle &= \langle \text{phys}'_1 | \left\{ \int d^3x \text{tr}[\vec{\Pi}'^2(\vec{x}) + \vec{B}'^2(\vec{x})] + \right. \\ &+ \left. \int d^2x_\perp \int dx_3 \int dy_3 \sum_{\substack{pq \\ p \neq q}} \frac{G'_{\perp pq}(\vec{x}_\perp, x_3) G'_{\perp pq}(\vec{x}_\perp, y_3)}{\left[\frac{2\pi}{L} n + g (A'_{3,q}(\vec{x}_\perp) - A'_{3,p}(\vec{x}_\perp)) \right]^2} e^{\frac{2\pi i}{L} n(x_3 - y_3)} \right\} | \text{phys}'_2 \rangle \end{aligned} \quad (2.15)$$

$$[A'_i(\vec{x}), \Pi'_j(\vec{x})] = \begin{cases} i\delta_{ij} \delta^{ab} \delta^{(3)}(\vec{x} - \vec{y}) & \text{for } i = 1, 2 \\ i\delta_{ij} \delta^{ab} \delta^{(2)}(\vec{x}_\perp - \vec{y}_\perp) & \text{for } i = 3 \end{cases}. \quad (2.16)$$

$\vec{B}'(\vec{x})$ is (2.2) with primed replacing unprimed variables as in $G'_\perp(\vec{x}) := \vec{D}'_\perp(\vec{x}) \cdot \vec{\Pi}'_\perp(\vec{x})$, and the Green's function of D_3 has been given explicitly in the “Coulomb” part, i.e. the last line of H . $G'_{\perp pq}$ is the (pq) entry of the matrix G'_\perp , and $A'_{3,p}$ the (pp) entry of the diagonal matrix A'_3 . The sum goes over all labels (pqn) for which the denominator is

nonzero. In (2.14) it was taken into account that a change of variables induces a Jacobian $\mathcal{J}[A'_3]$ in the Hilbert space measure³

$$\begin{aligned} \int \mathcal{D}A_3 &\rightarrow \int \mathcal{D}A'_3 \mathcal{J}[A'_3] = \int \mathcal{D}A'_3 \exp \delta^{(2)}(\vec{0}_\perp) \int d^2x_\perp \ln J[A'_3(\vec{x}_\perp)] , \\ \mathcal{J}[A'_3] &= \prod_{\vec{x}_\perp} J[A'_3(\vec{x}_\perp)] \quad , \quad J[A'_3(\vec{x}_\perp)] = \prod_{p>q} \sin^2 \frac{gL}{2} [A'_{3,q}(\vec{x}_\perp) - A'_{3,p}(\vec{x}_\perp)] , \end{aligned} \quad (2.17)$$

where $J[A'_3(\vec{x}_\perp)]$ is the Haar measure of $SU(N)$ for the case that the integrand depends only on the invariants. The measure in the Schrödinger representation of the Hilbert space of “radial wave functions” $|\text{phys}'\rangle$ is again $\int \mathcal{D}A'$, and $|\text{phys}'\rangle$ vanishes at the zeroes of the Jacobian,

$$|\text{phys}'\rangle = 0 \quad \forall \quad \mathcal{J}[A'_3] = 0 \quad , \quad \text{i.e.} \quad \frac{gL}{2\pi} [A'_{3,q}(\vec{x}) - A'_{3,p}(\vec{x}_\perp)] \in Z \quad \text{for some } \vec{x}_\perp, p \neq q . \quad (2.18)$$

The importance of this boundary condition in field space has been stressed recently [17] and is a remnant of the fact that $A'_3(\vec{x}_\perp)$ is an angle variable whose extraction from the Polyakov loop (2.11) is not unique.

Because of the occurrence of the zero modes A'_3, Π'_3 , a residual Gauß's law

$$\int dx_3 G'_\perp{}^{a_0}(\vec{x}) |\text{phys}'\rangle = 0 \quad \forall a_0 \quad (2.19)$$

survives which can be interpreted as eliminating all colour neutral, longitudinal gluons moving in the (x_1, x_2) -plane. It can be solved as above [3], but this is not necessary in what follows.

The residual gauge transformations can either be constructed by explicit transformation of (2.5) to the transformed physical Hilbert space [18]

$$\Omega'[\beta] |\text{phys}'\rangle = \sqrt{\mathcal{J}[A'_3]} U \Omega[\beta] U^\dagger \frac{1}{\sqrt{\mathcal{J}[A'_3]}} |\text{phys}'\rangle , \quad (2.20)$$

or – as sketched here – by an inspection [3] of the freedoms in the solution (2.12) of equation (2.9) defining $A'_3(\vec{x}_\perp)$, the eigenphases of the Polyakov loop (2.11).

$\Omega'[\beta] \vec{A}'(\vec{x}) \Omega^\dagger[\beta]$ again has to obey the gauge condition (2.10), so that the freedom to perform gauge transformations which mix diagonal and offdiagonal components of the fields \vec{A}' is removed. Together with the demand for periodicity of the gauge transformations (2.6), this yields in the generic case

$$\Omega'[\beta] \vec{A}'(\vec{x}) \Omega^\dagger[\beta] = \tilde{V}' \vec{A}'(\vec{x}) \quad : \quad \tilde{V}' = R \exp i \left[\frac{2\pi}{L} \vec{n} \cdot \vec{x} + \beta_{\text{period}}(\vec{x}_\perp) \right] . \quad (2.21)$$

$\beta_{\text{period}}(\vec{x}_\perp)$ are arbitrary gauge functions periodic in \vec{x}_\perp in a traceless diagonal matrix, \vec{n} a vector consisting of diagonal traceless matrices having integer entries, and R a member of the N dimensional representation of the permutation group \mathcal{S}_N , the Weyl (reflection)

³A normalization constant due to the integration over unphysical degrees of freedom has been dropped.

group of $SU(N)$, which changes the order of the entries in a diagonal matrix. The residual gauge group is therefore

$$G' := [U(1)]^{N-1} \times \mathcal{S}_N \quad , \quad (2.22)$$

and (as can be shown explicitly [18])

$$\Omega'[\beta] = \exp \left\{ -i \int d^3x \, 2\text{tr} \left[-\frac{2\pi}{gL} \left(\vec{\Pi}' \cdot \vec{D}' \right) (\vec{n} \cdot \vec{x}) + \left(\vec{D}' \cdot \vec{\Pi}' \right) \beta_{\text{period}}(\vec{x}_\perp) \right] \right\} \Omega_R \quad , \quad (2.23)$$

where Ω_R is the operator generating the Weyl permutations.

When the Polyakov loop has degenerate eigenvalues on some line $(\vec{x}_\perp, [0; L])$, the residual gauge group is larger on this closed loop in the x_3 -direction, namely $G' = [U(1)]^{m-1} \otimes_{i=1}^{m-1} SU(\alpha_i) \times \mathcal{S}_N$ [5, 19] for m different eigenvalues, each of degeneracy α_i . At these points, the Jacobian (2.17) of the coordinate transformation in field space vanishes, and hence the gauge fixing procedure is not defined at all. Sometimes, it is argued [8] that these nongeneric points in configuration space are mere coordinate singularities and don't have any gauge invariant meaning, especially since their set has zero measure and is in addition suppressed because the Coulomb part in the Hamiltonian (2.15) yields infinite energy. The arguments given in the introduction show that this may indeed be misleading and not render the properties of low-energy QCD known: Neglecting such configurations in a completely gauge field formulation, no winding number changing processes can be described.

One may nonetheless perform the gauge fixing in the above way for all points on T^3 for which the transformation $\tilde{U}(\vec{x})$ is single-valued and the Jacobian (2.17) is nonzero. All integrals are then understood to exclude these singular loops and so go over a manifold T_R^3 which may have holes. Then (2.21/2.23) gives the residual gauge transformations on $T_R^3 = T_R^2 \times S_{x_3}^1$, of which the displacements of the colour neutral fields

$$\Omega'[\beta] \vec{A}'_p(\vec{x}) \Omega'^\dagger[\beta] = \vec{A}'_p(\vec{x}) + \frac{2\pi}{gL} \vec{n} \quad , \quad \Omega'[\beta] \vec{\Pi}'_p(\vec{x}) \Omega'^\dagger[\beta] = \vec{\Pi}'_p(\vec{x}) \quad (2.24)$$

affect only their zero modes and will be of importance in what follows. Offdiagonal fields are rotated by \tilde{V}' .

The freedom to perform periodic, colour neutral gauge transformations in \vec{x}_\perp -direction is void in the transformed physical Hilbert space because of the residual Gauß's law (2.19), cf. (2.23). Note finally that the modified axial gauge lies in the class of abelian projection gauges [2]: The nonabelian $SU(N)$ gauge freedom is reduced to leave only $N - 1$ "electrodynamics".

3 Large and Residual Gauge Transformations

Before gauge fixing, the following quantum mechanical derivation for the occurrence of the vacuum- ϑ -angle can be given [20].

Physical states are annihilated by Gauß's law, the generator of infinitesimal gauge transformations, and hence are invariant under small gauge transformations (winding

number zero). The integral over the Chern–Simons three-form (zero component of the vector K^μ (1.1)),

$$W[A] := \frac{g^2}{16\pi^2} \int d^3x \epsilon^{ijk} \text{tr}[F_{ij} A_k + \frac{2i}{3} g A_i A_j A_k] , \quad (3.1)$$

serves as detector for the winding number of a gauge transformation, distinguishing small and large ones. It commutes with Gauß's law and therefore is invariant under small, but changes under large ones

$$\Omega[\beta]W[A]\Omega^\dagger[\beta] = W[A] + \nu(\tilde{V}) + \frac{ig}{8\pi^2} \int d^3x \epsilon^{ijk} \partial_i \text{tr}[A_j (\partial_k \tilde{V}^\dagger) \tilde{V}] \quad (3.2)$$

by the (additive) winding number

$$\nu(\tilde{V}) := \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{tr}[(\tilde{V} \partial_i \tilde{V}^\dagger)(\tilde{V} \partial_j \tilde{V}^\dagger)(\tilde{V} \partial_k \tilde{V}^\dagger)] \in Z . \quad (3.3)$$

The surface term in (3.2) vanishes because of the periodicity condition (2.6). This is closely related to the vanishing of all the total colour charges in the box [18, 21]

$$Q^a | \text{phys} \rangle := \int d^3x (f^{abc} \vec{A}^b(\vec{x}) \cdot \vec{\Pi}^c(\vec{x})) | \text{phys} \rangle = 0 \quad \forall a . \quad (3.4)$$

Since $\Omega[\beta]W[A]\Omega^\dagger[\beta]$ is in the physical Hilbert sector,

$$[Q^a, [Q^a, \Omega[\beta]W[A]\Omega^\dagger[\beta]]] | \text{phys} \rangle \stackrel{!}{=} 0 = \frac{ig}{8\pi^2} \int d^3x \epsilon^{ijk} \partial_i \text{tr}[A_j (\partial_k \tilde{V}^\dagger) \tilde{V}] | \text{phys} \rangle \quad (3.5)$$

$\Omega[\beta]$ therefore leaves physical states invariant only up to a phase into which – besides the winding number – the famous vacuum- ϑ -angle enters as new, hidden parameter:

$$\Omega[\beta] | \text{phys} \rangle = e^{i\vartheta\nu(\tilde{V})} | \text{phys} \rangle \quad (3.6)$$

If one assumed that all residual gauge transformations after the elimination of Gauß's law were large, one would require from (2.23)

$$\Omega'[\beta] | \text{phys}' \rangle = e^{i\vec{\vartheta} \cdot \vec{n}} | \text{phys}' \rangle \quad (3.7)$$

where $\vec{\vartheta}$ are $3(N-1)$ independent parameters in a vector of diagonal traceless matrices. This is obviously in conflict with the occurrence of only *one* free parameter before gauge fixing. Large and residual gauge transformations can therefore not be identical.

The correct procedure to identify large gauge transformations after gauge fixing is to start from the winding number detector (3.1) in the modified axial gauge, where \vec{A} is replaced by \vec{A}' , and investigate under which of the residual gauge transformations it changes by an integer:

$$\begin{aligned} \Omega'[\beta]W[A']\Omega'^\dagger[\beta] &\stackrel{!}{=} W[A'] + \nu(\tilde{V}) \\ &= W[A'] + n(\tilde{V}') + \frac{ig}{8\pi^2} \int_{T_R^3} d^3x \epsilon^{ijk} \partial_i \text{tr}[A'_j (\partial_k \tilde{V}'^\dagger) \tilde{V}'] \end{aligned} \quad (3.8)$$

Since $\vec{\partial}\tilde{V}'$ is abelian ($\vec{\partial}R = 0$) in T_R^3 , $n(\tilde{V}') = 0$ and hence the surface integral must be nonvanishing, which suggests that regularity or boundary conditions of the fields have to be changed by the UGFT.

A valuable clue towards an identification of the relevant degrees of freedom yielding nonzero winding number is the fact that in the transformed physical Hilbert space, only the neutral components of the transformed total colour charges annihilate physical states

$$Q'^{a_0} | \text{phys}' \rangle := \int_{T_R^3} d^3x \left[f^{a_0bc} \vec{A}_\perp^b(\vec{x}) \cdot \vec{\Pi}_\perp^c(\vec{x}) \right] | \text{phys}' \rangle \stackrel{!}{=} 0 \quad \forall a_0, \quad (3.9)$$

since offdiagonal global transformations in general violate the gauge condition (2.10). Indeed, condition (3.9) can also be derived by performing the UGFT of (3.4) [18] or by integrating (2.19) over \vec{x}_\perp and using periodicity and continuity for the longitudinal colour neutral fields $\vec{A}_{\perp,p}^n$ which will later be demonstrated to hold.

The analogous calculation to (3.5) eliminates only the offdiagonal components of the surface integral

$$\int_{T_R^3} d^3x \epsilon^{ijk} \partial_i \text{tr}[A'_j(\partial_k \tilde{V}'^\dagger) \tilde{V}'] | \text{phys}' \rangle = \frac{1}{2} \int_{T_R^3} d^3x \epsilon^{ijk} \partial_i [A_j'^{a_0}[(\partial_k \tilde{V}'^\dagger) \tilde{V}']^{a_0}] | \text{phys}' \rangle \quad (3.10)$$

and hence suggests to investigate boundary conditions and continuity properties of the colour neutral fields after the UGFT in order to identify the large gauge transformations.

All variables are still periodic in x_3 since \tilde{U} (2.12) is explicitly periodic and continuous if $A_3(\vec{x})$ was. On the other hand, periodicity in \vec{x}_\perp can in general not be maintained for two reasons.

Although the eigenvalues $\exp igLA'_3(\vec{x}_\perp)$ of the Polyakov loop (2.11) are periodic and continuous, the last term in the explicit solution \tilde{U} (2.12) makes it necessary to take its logarithm in order to define $A'_3(\vec{x}_\perp)$. As has been demonstrated in [3], $A'_3(\vec{x}_\perp)$ may be chosen to be continuous on all of T^3 , but then the boundary conditions of the phases will in general be changed to⁴

$$A'_3(\vec{x}^{(i)} + L\vec{e}_i) - A'_3(\vec{x}^{(i)}) = \frac{2\pi}{gL} \varepsilon^{ij} m_j \quad \text{for } i = 1, 2 \quad (3.11)$$

where ε^{ij} is the totally antisymmetric unit tensor of second rank, $i, j = 1, 2$ and m_i a diagonal, traceless matrix with integer entries. The mapping $\exp igLA'_3(\vec{x}_\perp) : T_R^2 \rightarrow [U(1)]^{N-1}$ decomposes into topologically distinct classes labelled by the two winding numbers $m_i \in \mathbb{Z}^{N-1}$.

Furthermore, the diagonalization matrix $e^{ig\Delta}$ (2.13) is determined up to right multiplication with an element of the equivalence class of $\exp igLA'_3(\vec{x}_\perp)$ [5, 19], i.e. $e^{ig\Delta(\vec{x}_\perp)} \in SU(N)/G'$ ($\vec{x}_\perp \in T_R^2$) lies in the coset. This suggests the occurrence of additional singularities in the transverse fields \vec{A}_\perp whenever $(\partial_1 \partial_2 - \partial_2 \partial_1) e^{ig\Delta(\vec{x}_\perp)} \neq 0$. Nonetheless, such points have been excluded by the choice of T_R^3 . The diagonalization can be chosen to be continuous on T_R^3 , but in general will not be periodic because the mapping

⁴ This definition of m_i differs from the one given in [3].

$SU(N)/G' \rightarrow T_R^2$ decomposes into topologically distinct classes or the first homotopy group of $[U(1)]^{N-1}$ is Z^{N-1} .

$$e^{ig\Delta(\vec{x}^{(i)}+L\vec{e}_i)} = e^{ig\Delta(\vec{x}^{(i)})} h(\vec{x}^{(i)}) \quad , \quad h(\vec{x}^{(i)}) \in G' \quad (3.12)$$

So it proves again impossible to find a gauge fixing procedure to the modified axial gauge (2.10) which is both periodic and continuous for all gauge field configurations. If one wants to preserve continuity, the boundary conditions change from (2.6) to

$$\begin{aligned} U \left(\vec{A}(\vec{x}^{(i)} + L\vec{e}_i) - \vec{A}(\vec{x}^{(i)}) \right) U^\dagger &= \tilde{U} \vec{A}'(\vec{x}^{(i)} + L\vec{e}_i) - \tilde{U} \vec{A}'(\vec{x}^{(i)}) = 0 \\ \implies \vec{A}'(\vec{x}^{(i)} + L\vec{e}_i) &= u^{(i)} \left(\vec{A}'(\vec{x}^{(i)}) + \frac{i}{g} \vec{\partial} \right) u^{(i)\dagger} \\ \vec{\Pi}'(\vec{x}^{(i)} + L\vec{e}_i) &= u^{(i)} \vec{\Pi}'(\vec{x}^{(i)}) u^{(i)\dagger} \quad , \quad \vec{V}(\vec{x}^{(i)} + L\vec{e}_i) = u^{(i)} \vec{V}(\vec{x}^{(i)}) u^{(i)\dagger} \quad , \end{aligned} \quad (3.13)$$

where

$$u^{(i)}(\vec{x}^{(i)}) = \begin{cases} e^{\frac{2\pi i}{L} x_3 \varepsilon^{ij} m_j} h(\vec{x}^{(i)}) & \text{for } i = 1, 2 \\ \mathbf{1} & \text{for } i = 3 \end{cases} \quad . \quad (3.14)$$

This shows that both the residual gauge transformations and the longitudinal colour neutral fields in (2.19) again have to be periodic and continuous, justifying (2.21) and (3.9), while the transverse colour neutral fields are periodic up to a shift, as the rest of the fields up to a rotation about axes in colour space corresponding to diagonal generators.

The winding numbers of the above mappings do not interfere since different diagonalization matrices $e^{ig\Delta}$ cannot change the eigenvalues of the Polyakov loop and vice versa and are the diagonal matrices

$$\begin{aligned} m_1 &= \frac{g}{2\pi} \int_{T_R^2} dx_3 dx_2 \quad \partial_2 A'_{3,p}(\vec{x}_\perp) = \frac{g}{2\pi L} \int_{T_R^3} d^3x \quad B'_{1,p}(\vec{x}) \\ m_2 &= -\frac{g}{2\pi} \int_{T_R^2} dx_3 dx_1 \quad \partial_1 A'_{3,p}(\vec{x}_\perp) = \frac{g}{2\pi L} \int_{T_R^3} d^3x \quad B'_{2,p}(\vec{x}) \\ m_3 &= \frac{g}{2\pi} \oint_{\partial[T_R^2]} d\vec{s}_\perp \cdot \vec{A}'_{\perp,p}(\vec{x}) - \frac{g}{2\pi} \oint_P d\vec{s}_\perp \cdot \vec{A}'_{\perp,p}(\vec{x}) = \frac{g}{2\pi L} \int_{T_R^3} d^3x \quad b'_{3,p}(\vec{x}) \quad . \end{aligned} \quad (3.15)$$

(Note that $b'_{3,p} := \partial_1 A'_{2,p} - \partial_2 A'_{1,p} \neq B'_{3,p}$ and $\text{tr}[\vec{m}] = 0$.) \vec{m} is invariant under residual gauge transformations (2.21), except for permutation of entries, and hence an observable. Together with (3.8/3.10), this shows that only configurations with nonzero (abelian) magnetic fluxes through the box can have mirror configurations which differ by a large gauge transformation of winding number

$$\nu(\tilde{V}') = \frac{1}{2} \text{tr}[\vec{n} \cdot \vec{m}] \quad . \quad (3.16)$$

These configurations describe abelian magnetic vortices whose total magnetic flux obeys the Dirac quantization condition. This is again reminiscent of the Dirac monopole as discussed in the introduction.

With respect to all other gauge configurations, the gauge is therefore completely fixed (up to small gauge transformations connected to the residual Gauß's law (2.19)). The opportunity not only for small, but also for large gauge transformations has been eliminated for all zero magnetic flux configurations in the modified axial gauge (2.10).

4 Discussion

That the resolution of Gauß's law in the space of transformed physical states eliminated indeed not only the opportunity for small, but also for large gauge transformations for many configurations, can be traced back to the fact that in general the UGFT will not leave $W[A]$ (3.1) invariant, but changes it by

$$U W[A] U^\dagger = W[A'] + \nu(\tilde{U}) + \frac{ig}{8\pi^2} \int d^3x \epsilon^{ijk} \partial_i \text{tr}[A_j (\partial_k \tilde{U}^\dagger) \tilde{U}] . \quad (4.1)$$

One should note that $\tilde{U}[A_3]$ depends on the unphysical variables whose conjugate momenta have been eliminated in the physical Hilbert space. One can show [18] that there exists indeed no solution $\tilde{U}[A_3]$ to the gauge fixing procedure (2.9/2.10) which is periodic and continuous in all directions, as well as “small”, i.e. which would yield

$$U W[A] U^\dagger = W[A'] \quad (4.2)$$

for all field configurations. If (4.2) would hold, every point in the physical Hilbert space had mirror points of the same physics which can be reached by large gauge transformations. This is the case for the axial and Coulomb gauge representation of QED [15], where the UGFT leaves the zero modes of the fields unchanged, which are the winding number detectors. Therefore with the resolution of Gauß's law, only the unitary operator of small gauge transformations is reduced to the identity in the physical sector. In contradistinction, in modified axial gauge QCD one has to allow for singular or nonperiodic gauge configurations to implement large gauge transformations.

Although the set they form may be of zero measure, these configurations are indispensable since only their occurrence can be connected to the existence of the vacuum- ϑ -angle which is known to be relevant at least in semiclassical approximations.

On the other hand, several questions need clarification. Since with (2.15/3.15/3.13), $[H', \vec{m}_\perp] = 0$, the total magnetic fluxes in \vec{x}_\perp -directions are constants of motion. Choosing furthermore the fields to be continuous in T^3 , the vortex configurations are described by fields A'_3 which contain at least $2\vec{m}_\perp$ closed loops in x_3 -direction at which the Jacobian $\mathcal{J}[A'_3]$ (2.17) is zero, yielding zero wavefunction (2.18). One may therefore rule out tunneling processes between configurations with $\vec{m}_\perp \neq 0$, which are connected by displacements $\vec{n}_\perp \neq 0$, although they are connected by large gauge transformations (3.16), and restrict oneself to the only sector of nonzero probability amplitude, $\vec{m}_\perp = 0$.

The argument is less stringent for m_3 since $-[H', m_3]$ being nonzero – it is not time independent. Still, its equation of motion is undecided whenever the denominator of the “Coulomb term” in (2.15) becomes zero in time evolution. If time evolution between two configurations connected by displacements $n_3 \neq 0$ is again continuous for A'_3 , then it crosses – as above – at $2n_3$ points in time configurations with zero Jacobian. At these points, the (radial) wavefunction vanishes, which seemingly again forbids tunneling between $m_3 \neq 0$, $n_3 \neq 0$ -configurations despite of their being partly connected by large gauge transformations.

One might therefore be tempted to conclude that – although large gauge transformations exist – the vacuum- ϑ -angle becomes irrelevant because every (semiclassical) tunneling process between configurations connected by large gauge transformations is suppressed

by the Jacobian. On the other hand, Singer [10] shows that the points in configuration space at which a complete gauge fixing procedure is singular (i.e. at the Gribov horizon) will necessarily have zero Jacobian. The Gribov horizon is then just defined as this manifold of gauge configurations which forms an impenetrable barrier for field configurations in time evolution. Only there, an identification of points which differ by a large gauge transformation is possible and incorporates the physics connected to the vacuum- ϑ -angle. This is reminiscent of the picture of a pendulum at turning point [9, 13].

It is well known that before gauge fixing there exists a semi-classical configuration (the instanton) of finite action and energy which extrapolates between physically equivalent configurations connected by large gauge transformations for every winding number. Such semiclassical solutions to the Heisenberg equations of motion resulting from (2.15) must be found after the UGFT, too, now running between two points at the Gribov Horizon. Maybe dynamics indeed forbids one to interpolate in time between certain configurations $\vec{m} \neq 0$, $\vec{n} \neq 0$. Such a decoupling would also provide a solution to the problem of half-integer winding numbers occurring in (3.16) for $SU(N)$, $n > 2$, whenever \vec{m} and \vec{n} are members of overlapping, but not identical subalgebrae of the Cartan algebra.

5 Conclusions

Gauge fixing to the modified axial gauge, in which the eigenphases of the Polyakov loops in x_3 -direction are kept as dynamical variables, yielded with the Hamiltonian (2.15) unique vector potentials at the generic points in the physical sector of configuration space. The *nongeneric* points are defined as the ones at which the coordinate transformation (2.9) in field space which is part of the gauge fixing process (UGFT) [3, 15] becomes singular or non-periodic in \vec{x}_\perp . The unitary operator Ω' (2.23) implementing residual gauge transformations \tilde{V}' (2.21) has been given in the physical Hilbert space, i.e. in terms of unconstrained variables. Large gauge transformations exist only for the nongeneric set of configurations.

The winding number of a residual gauge transformation was – due to these singularities – not given by a field-independent volume integral (3.3) but by a surface integral (3.8/3.10/3.16) which measured the total colour neutral magnetic fluxes (3.15) in the directions in which the displacements of the zero modes of the colour neutral gluons (2.24) act on the fields. Therefore, the existence of abelian magnetic vortex configurations was necessary to allow for large gauge transformations in the modified axial gauge.

The occurrence of singularities and changes of boundary conditions proved to be a natural outcome of the UGFT (2.9/2.12). Besides ambiguities in the diagonalization of the Polyakov loop which had already been found in [5, 19], this was attributed to a projection of its eigenphases $A'_3(\vec{x}_\perp)$ at the boundaries of the box onto different Riemann sheets [3].

Vortex configurations with nonzero magnetic flux in the \vec{x}_\perp -directions are connected to the problems in extracting the eigenphases. Taking into account the dynamics of the quantum mechanical system, it seems that their wave function is zero because of the Jacobian (2.17/2.18) arising from the coordinate transformation of the UGFT to curvilinear coordinates in the physical Hilbert space. Nonetheless, the vortex configurations with

nonzero magnetic flux in x_3 -direction, arising from ambiguities in the diagonalization of the Polyakov loop, are admissible. For them, time evolution between configurations differing by a large gauge transformation seems to be forbidden, again because of the Jacobian. That it appears to prohibit any tunneling, is a quantum mechanical phenomenon and lies beyond semiclassical treatment. On the other hand, points at which the Jacobian is zero were excluded from T^3 in the UGFT, and a semiclassical approximation should render the well-known results of instanton calculations.

A detailed, more physically motivated study of the winding number changing processes and their possible suppression by (or circumvention of) the Jacobian is under way. The question to what extent the magnetic vortices are localized in a semiclassical solution of the dynamics is presently addressed, too. It is also investigated whether these vortices may serve as relevant configurations for the description of low energy properties of QCD, as in the abelian projection gauges [1, 2]. The hope is that – having understood these issues – one may perform suitable approximations in the completely gauge fixed Hamiltonian framework in order to improve our insight into the low energy sector of QCD, which is generally attributed to topological “nontrivialities”.

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